A note on Mott-Smith's solution of the Boltzmann equation for a shock wave

By AKIRA SAKURAI

*Department of Applied Mathematics, Weizmann Institute of Science, Israel**

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SUMMARY

After a modification, the interpolation formula of Mott-Smith (1951) for the shock wave problem is found to be a solution of the Boltzmann equation at large Mach number in a finite region of molecular velocity space. This modification gives a unique determination of the shock wave thickness, removing the ambiguity for this in Mott-Smith's formula.

Since the classical paper by Becker, it has been believed that the problem of shock wave structure must be considered in the light of the kinetic theory of gases. Many attempts have been made to find the solution of the basic Boltzmann equation, especially in the case of weak shock waves where the solution might be considered to be not so different from the Maxwell distribution. However, according to recent experiments on weak shock waves by Talbot & Sherman (1956), measured shock profiles are in rather better agreement with the predictions of the Navier–Stokes equations than with the results of the approximate solutions of the Boltzmann equation for weak shock waves. An approximate solution for a strong shock was given by Mott-Smith (1951). He considered that the molecular velocity distribution in a strong shock wave must be bimodal because of the effects of bounding supersonic and subsonic regions where the velocity distributions are of Maxwell type with different physical constants. Since the series of monocentric functions used in the approximate solutions for weak shock waves are inappropriate to represent such a bimodal distribution, Mott-Smith's approximate solution consisting of a sum of the two Maxwell distributions seems to be a more reasonable solution of the shock wave problem.

If we write $f(\mathbf{c}, x)$ for the distribution function of molecular velocity $\mathbf{c}$ in a plane shock wave whose normal is in the x direction, Mott-Smith's approximation $f^{(0)}$ has the form

$$f^{(0)} = \nu_\alpha(x) f_\alpha + \nu_\beta(x) f_\beta, \tag{1}†$$

where f_α, f_β signify the Maxwell distributions in the uniform super and

* Now at Tokyo Electrical Engineering College, Kanda, Tokyo.

† The definitions of f_α, f_β differ from Mott-Smith's by the factors $\nu_\alpha(x)$, $\nu_\beta(x)$ respectively.

subsonic regions respectively and are given by

$$f_\alpha = n_\alpha \left(\frac{m}{2\pi k T_\alpha}\right)^{3/2} \exp\left\{-\frac{m}{2kT_\alpha}(\mathbf{c}-\mathbf{i}u_\alpha)^2\right\}, \tag{2}^*$$

$$f_\beta = n_\beta \left(\frac{m}{2\pi k T_\beta}\right)^{3/2} \exp\left\{-\frac{m}{2kT_\beta}(\mathbf{c}-\mathbf{i}u_\beta)^2\right\}, \tag{3}^*$$

where n_α, n_β are the number densities, T_α, T_β are the temperatures, $\mathbf{i}u_\alpha$, $\mathbf{i}u_\beta$ are the stream velocities and m is the mass of the molecule.

To determine the functions $\nu_\alpha(x)$, $\nu_\beta(x)$, Mott-Smith utilized the transport equation for an arbitrary function $\Phi(\mathbf{c})$

$$\frac{\partial}{\partial x}\left(\int u\Phi(\mathbf{c})f\,d\mathbf{c}\right) = \iiint \{\Phi(\mathbf{c}') - \Phi(\mathbf{c})\} f f_1 g\, d\Omega d\mathbf{c}_1\, d\mathbf{c},$$

which is equivalent to an averaging of the one-dimensional Boltzmann equation

$$L(f) \equiv u\frac{\partial f}{\partial x} - \iint (f'f_1' - ff_1) g\, d\Omega d\mathbf{c}_1 = 0, \tag{4}$$

in the velocity-space $\mathbf{c}$ with the weight $\Phi(\mathbf{c})$, where g is the magnitude of the relative velocity $\mathbf{g} = \mathbf{c}_1 - \mathbf{c}$ of a colliding pair; f_1, f', f_1' represent $f(\mathbf{c}_1, x)$, $f(\mathbf{c}', x)$, $f(\mathbf{c}_1', x)$; and $\mathbf{c}'$, $\mathbf{c}_1'$ denote the velocities of a colliding pair after collision; and $d\Omega$ is a differential cross-section. Taking $\Phi(\mathbf{c}) = u^2, u^3$, it was found that

$$\nu_\alpha(x) = \nu(-x), \qquad \nu_\beta(x) = \nu(x), \qquad \nu(x) \equiv \tfrac{1}{2}(1 + \tanh 2x/X), \tag{5}$$

where X is a function of Mach number $M = u_\alpha/c_\alpha$ (c_α is the velocity of sound in the supersonic flow) and gives the thickness of the shock wave.

Now the form of the function $X(M)$ depends on the choice of the function $\Phi(\mathbf{c})$ but the forms of the functions $\nu_\alpha(x)$, $\nu_\beta(x)$ are always the same for any $\Phi(\mathbf{c})$ and are given by (5). The purpose of the present note is to show that if we take a special form for the function $X(M)$, the equation

$$f^{(0)} = \nu(-x)f_\alpha + \nu(x)f_\beta \tag{6}$$

satisfies directly the Boltzmann equation (4) at large M for a finite, fixed value of c. This choice of $X(M)$ may be useful in removing the ambiguity about the function $X(M)$ caused by its dependence on $\Phi(\mathbf{c})$.

Substituting from (6) into (4) and utilizing the relations

$$\frac{d}{dx}\nu(x) = -\frac{d}{dx}\nu(-x) = \frac{4}{X}\nu(x)\nu(-x), \tag{7}$$

$$f_\alpha' f_{\alpha 1}' = f_\alpha f_{\alpha 1}, \qquad f_\beta' f_{\beta 1}' = f_\beta f_{\beta 1}, \tag{8}$$

we get

$$L(f^{(0)}) = \nu(x)\nu(-x)L_c(f^{(0)}), \tag{9}$$

where

$$L_c(f^{(0)}) = \frac{4}{X}u(f_\beta - f_\alpha) - \iint (f_\alpha' f_{\beta 1}' + f_{\alpha 1}' f_\beta' - f_\alpha f_{\beta 1} - f_{\alpha 1} f_\beta) g\, d\Omega d\mathbf{c}_1.$$

It is to be noted that the dependence on x and $\mathbf{c}$ is separated in (9), and $L(f^{(0)}) \to 0$ as $x \to \pm\infty$ since $\nu(x)\nu(-x) = \frac{1}{4}\operatorname{sech}^2(2x/X)$. The term $L_c(f^{(0)})$

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in (9) will be shown to become small for a special $X(M)$ when M becomes large for finite, fixed c.

To do this, we shall first transform $L_c(f^{(0)})$. It follows from (8) that

$$f'_\alpha f'_{\beta 1} = f_\beta f_{\beta 1}\frac{f'_\alpha}{f'_\beta}, \qquad f'_{\alpha 1} f'_\beta = f_\beta f_{\beta 1}\frac{f'_{\alpha 1}}{f'_{\beta 1}},$$

and hence

$$\iint (f'_\alpha f'_{\beta 1} + f'_{\alpha 1} f'_\beta - f_\alpha f_{\beta 1} - f_{\alpha 1} f_\beta) g\, d\Omega d\mathbf{c}_1$$
$$= \iint f_\beta f_{\beta 1}\left(\frac{f'_{\alpha 1}}{f'_{\beta 1}} + \frac{f'_\alpha}{f'_\beta} - \frac{f_{\alpha 1}}{f_{\beta 1}} - \frac{f_\alpha}{f_\beta}\right) g\, d\Omega d\mathbf{c}_1. \quad (10)$$

Putting $\qquad \mathbf{p}_\beta = (m/2kT_\beta)^{1/2}(\mathbf{c} - \mathbf{i}u_\beta), \qquad \mathbf{g}_\beta = \mathbf{p}_{\beta 1} - \mathbf{p}_\beta, \qquad (11)$

the right-hand side of equation (10) becomes

$$\left(\frac{2kT_\beta}{m}\right)^{1/2} n_\beta f_\beta J_\beta\left(\frac{f_\alpha}{f_\beta}\right),$$

where $$J_\beta(h) \equiv \pi^{-3/2} \iint \exp(-p_{\beta 1}^2)(h'_1 + h' - h_1 - h) g_\beta\, d\Omega d\mathbf{p}_{\beta 1}. \quad (12)$$

Then $L_c(f^{(0)})$ becomes

$$L_c(f^{(0)}) = -f_\alpha u_\alpha\left[\frac{4}{X}\left(1 + \frac{1}{u_\alpha}\left(\frac{2kT_\alpha}{m}\right)^{1/2}\right)\left(1 - \frac{f_\beta}{f_\alpha}\right) + \right.$$
$$\left. + \frac{1}{u_\alpha}\left(\frac{2kT_\beta}{m}\right)^{1/2} n_\beta \frac{f_\beta}{f_\alpha} J_\beta\left(\frac{f_\alpha}{f_\beta}\right)\right], \quad (13)$$

where $p_{\alpha x}$ is the x component of $\mathbf{p}_\alpha = (m/2kT_\alpha)^{1/2}(\mathbf{c} - \mathbf{i}u_\alpha)$.

From (2) and (3), we have

$$\frac{f_\beta}{f_\alpha} J_\beta\left(\frac{f_\alpha}{f_\beta}\right) = \exp(p_\alpha^2 - p_\beta^2) J_\beta\{\exp(p_\beta^2 - p_\alpha^2)\}, \quad (14)$$

and utilizing the standard simplified form of the expression (12) (see, for example, Chapman & Cowling 1953),

$$J_\beta(h) = -K_0(\mathbf{p}_\beta) h(\mathbf{p}_\beta) - \int h(\mathbf{p}_{\beta 1}) K(\mathbf{p}_\beta, \mathbf{p}_{\beta 1})\, d\mathbf{p}_{\beta 1},$$

where $$K_0(\mathbf{p}_\beta) = \pi^{-3/2} \iint \exp(-p_\beta^2) g_\beta\, d\Omega d\mathbf{p}_{\beta 1}$$

and $K(\mathbf{p}_\beta, \mathbf{p}_{\beta 1})$ is a symmetric function of $\mathbf{p}_\beta$, $\mathbf{p}_{\beta 1}$, equation (14) reduces to

$$\frac{f_\beta}{f_\alpha} J_\beta\left(\frac{f_\alpha}{f_\beta}\right) = -K_0(\mathbf{p}_\beta) - \exp(p_\alpha^2 - p_\beta^2) \int \exp(p_{\beta 1}^2 - p_{\alpha 1}^2) K(\mathbf{p}_\beta, \mathbf{p}_{\beta 1})\, d\mathbf{p}_{\beta 1}. \quad (15)$$

For the case of the elastic sphere model of diameter σ, we have

$$K_0(\mathbf{p}_\beta) = \pi^{1/2}\sigma^2 F(p_\beta^2), \qquad F(x^2) = e^{-x^2} + \frac{2x^2 + 1}{x}\int_0^x e^{-t^2}\, dt, \quad (16)$$

and

$$\exp(p_\alpha^2 - p_\beta^2) \int \exp(p_{\beta 1}^2 - p_{\alpha 1}^2) K(\mathbf{p}_\beta, \mathbf{p}_{\beta 1})\, d\mathbf{p}_{\beta 1}$$

$$= \pi^{1/2}\sigma^2 \exp(p_\alpha^2 - p_\beta^2)\left\{k^4 F(p_\alpha^2) - \frac{2}{\pi}\int \exp(-p_{\alpha 1}^2 + \omega_\beta^2)\frac{1}{g_\beta}\, d\mathbf{p}_{\beta 1}\right\}, \quad (17)$$

where $\omega_\beta^2 = \{[\mathbf{p}_\beta, \mathbf{p}_{\beta 1}]/g_\beta\}^2, \qquad k = (T_\alpha/T_\beta)^{1/2}.$ (18)

Writing $u_{\alpha\beta} = (m/2kT_\beta)^{1/2}(u_\alpha - u_\beta),$

we have $\mathbf{p}_\beta = k\mathbf{p}_\alpha + \mathbf{i}u_{\alpha\beta}.$ (19)

Utilizing the Rankine-Hugoniot relation, k and $u_{\alpha\beta}$ become

$$\left.\begin{aligned} k &= (a-1)a^{-1/2}\frac{1}{M}\left(1+\frac{a-2}{M^2}\right)^{-1/2}\left(1-\frac{1}{aM^2}\right)^{-1/2}, \\ u_{\alpha\beta} &= \left(\frac{a-2}{2}\right)^{1/2}\left(1-\frac{1}{M^2}\right)\left(1+\frac{a-2}{M^2}\right)^{-1/2}\left(1-\frac{1}{aM^2}\right)^{-1/2}, \end{aligned}\right\} \quad (20)$$

where $a = 2\gamma/(\gamma-1)$, and γ is the ratio of specific heats.

When M becomes large, k becomes small while $u_{\alpha\beta}$ remains finite and approaches the value $\{(a-2)/2\}^{1/2}$. Accordingly $\mathbf{p}_\beta$ in (19) lies near $\mathbf{i}u_{\alpha\beta}$ provided p_α is finite and fixed, and then $F(p_\beta^2)$ in (16) can be expressed as

$$F(p_\beta^2) \sim F(u_{\alpha\beta}^2) + 2ku_{\alpha\beta}p_{\alpha x}F'(u_{\alpha\beta}^2) + k^2\{p_\alpha^2 F'(u_{\alpha\beta}^2) + 2u_{\alpha\beta}^2 p_{\alpha x}^2 F''(u_{\alpha\beta}^2)\} + \dots, \quad (21)$$

where we have used the equation

$$p_\beta^2 = u_{\alpha\beta}^2 + 2ku_{\alpha\beta}\, p_{\alpha x} + k^2 p_\alpha^2$$

and primes in F means differentiation with respect to p_β^2.

The integral in (17)

$$\frac{2}{\pi}\int \exp(-p_{\alpha 1}^2 + \omega_\beta^2)\frac{1}{g_\beta}\, d\mathbf{p}_{\beta 1}$$

will now be shown to be of the order of k^2. Utilizing the relations

$$g_\beta = kg_\alpha, \qquad d\mathbf{p}_\beta = k^3\, d\mathbf{p}_\alpha,$$

$$[\mathbf{p}_\beta, \mathbf{p}_{\beta 1}]/g_\beta = u_{\alpha\beta}[\mathbf{i}, \mathbf{g}_\alpha/g_\alpha] + k[\mathbf{p}_\alpha, \mathbf{g}_\alpha/g_\alpha],$$

and under the assumption of finite and fixed p_α, we get

$$\frac{2}{\pi}\exp(p_\alpha^2 - p_\beta^2)\int \exp(-p_{\alpha 1}^2 + \omega_\beta^2)\frac{1}{g_\beta}\, d\mathbf{p}_{\beta 1} \sim k^2 I(\mathbf{p}_\alpha) + \dots, \quad (22)$$

where $I(\mathbf{p}_\alpha) = \frac{2}{\pi}\exp(p_\alpha^2 - p_\beta^2)\int \frac{1}{g_\alpha}\exp(-p_\alpha^2 + u_{\alpha\beta}^2\sin^2\psi)\, d\mathbf{p}_{\alpha 1},$

and ψ denotes the angle between $\mathbf{g}_\alpha$ and $\mathbf{i}$.

Now from (20) and the Rankine–Hugoniot relations,

$$\left.\begin{aligned} \frac{1}{u_\alpha}\left(\frac{2kT_\alpha}{m}\right)^{1/2} &= \left(\frac{2(a-2)}{a}\right)^{1/2}\frac{1}{M},\\ \frac{1}{u_\alpha}\left(\frac{2kT_\beta}{m}\right)^{1/2} n_\beta &= n_\alpha\{2(a-2)\}^{1/2}A(M),\\ A(M) &\equiv \left(1-\frac{1}{aM^2}\right)^{1/2}\left(1+\frac{a-2}{M^2}\right)^{-1/2}, \end{aligned}\right\} \tag{23}$$

and equation (13) then finally becomes for finite and fixed p_α,

$$\begin{aligned} L_c(f^{(0)}) \sim -f_\alpha u_\alpha \Big[&\{(4/X) - \pi^{1/2}\sigma^2 n_\alpha\{2(a-2)\}^{1/2}A(M)F(u_{\alpha\beta}^2)\} \\ &+ k\{2(a-2)\}^{1/2}\left\{\frac{4}{X(a-1)}\left(1+\frac{a-2}{M^2}\right)^{1/2}\left(1-\frac{1}{aM^2}\right)^{1/2} - \right.\\ &\left. - 2\pi^{1/2}\sigma^2 n_\alpha A(M)u_{\alpha\beta}F'(u_{\alpha\beta}^2)\right\} p_{\alpha x} - k^2\pi^{1/2}\sigma^2 n_\alpha\{2(a-2)\}^{1/2}A(M)\times \\ &\times\{p_\alpha^2 F'(u_{\alpha\beta}^2) + 2u_{\alpha\beta}^2 p_{\alpha x}^2 F''(u_{\alpha\beta}^2) - I(\mathbf{p}_\alpha)\} + \dots\Big]. \end{aligned} \tag{24}$$

By equating the first term in (24) to zero, we can determine $X(M)$ from

$$\frac{4l}{X} = \left(\frac{a-2}{\pi}\right)^{1/2} A(M)F(u_{\alpha\beta}^2), \tag{25}$$

where l is the mean free path defined by

$$1/l = \surd 2\pi n_\alpha \sigma^2.$$

For the case of other molecular models, the second term in (15) is also likely to be small for large M and we may determine X from

$$\frac{4}{X} = n_\alpha\{2(a-2)\}^{1/2}A(M)K_0(\mathrm{i}u_{\alpha\beta}).$$

In the following table, values of l/X for $a = 5$ in (25) and for various Mach numbers are compared with Mott-Smith's values and the results of

M	l/X	$(l/X)_R$	$(l/X)_{u^3}$	$(l/X)_{u^2}$
∞	0·703	—	0·628	0·468
10	0·685	—	0·600	0·455
5	0·630	—	0·527	0·419
4	0·596	0·495	0·478	0·397
3	0·520	0·414	0·397	0·346
2·5	0·474	0·355	0·332	0·304

Rosen (1954), determined by a restricted variational method using an expression of type (1) as a test function. In the table $(l/X)_R$ denotes the results of Rosen, while $(l/X)_{u^2}$, $(l/X)_{u^3}$ give Mott-Smith's values for the cases $\Phi = u^2, u^3$ respectively.

When X is given by (25), (24) reduces to

$$L_c(f^{(0)}) \sim kL_1 + k^2L_2 + \dots,$$

where

$$L_1 = [2(a-2)]^{1/2}\frac{4}{X}\left\{\frac{1}{a-1}\left(1+\frac{a-2}{M^2}\right)^{1/2}\left(1-\frac{1}{aM^2}\right)^{1/2} - \right.$$
$$\left. - \sqrt{\frac{2}{\pi}}\frac{X}{4l}A(M)\,u_{\alpha\beta}\,F'(u_{\alpha\beta})\right\}p_{\alpha x},$$

$$L_2 = \frac{1}{l}\left(\frac{a-2}{\pi}\right)^{1/2} A(M)\{p_\alpha^2 F'(u_{\alpha\beta}^2) + 2u_{\alpha\beta}^2 p_{\alpha x}^2 F''(u_{\alpha\beta}^2) - I(\mathbf{p}_\alpha)\}.$$

This suggests that we may find a solution f in the form

$$f \sim f^{(0)} + k\phi^{(1)} + k^2\phi^{(2)} + \dots .$$

$\phi^{(1)}$ can be found easily and is given by

$$\phi^{(1)} = -(4/X)L_1\,\nu(-x)\log\nu(-x),$$

where we have used the boundary conditions for f, equations (2) and (3). Since $L_1 \propto p_{\alpha x}$ and the mean density $\bar{n}$ is given by $\bar{n} = \int f\,d\mathbf{c}$, $\phi^{(1)}$ does not contribute to $\bar{n}$ and, accordingly, not to the thickness. It seems, however, difficult to find $\phi^{(2)}$, $\phi^{(3)}$, ... successively, mainly because $\phi^{(1)}$, $\phi^{(2)}$, ... are of ordinary order of magnitude only in a finite region of $\mathbf{c}$ space which depends on k, and we need to estimate the integrals in which they are included over the whole of $\mathbf{c}$ space.

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